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COMMENT

Some observations on 'Spectral laws for the enstrophy cascade in two-dimensional turbulence'

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Abstract. We show that the β model cannot be used to describe the enstrophy cascade in two-dimensional fully developed turbulence, in contrast with some recent claims.

Moreover we point out that the scaling law for the energy spectrum cannot be deduced by the scaling of the velocity difference, because of the non-locality of the interactions in the Fourier space.

Shivamoggi (1990) has recently introduced a β model for the enstrophy cascade in two-dimensional (2D) turbulence. We want to point out that such a model is not consistent with global regularity properties for the 2D Euler equation in a bounded domain which are consequence of the vorticity conservation (Rose and Sulem 1978). Indeed, the following inequality can be obtained for the differences $\delta v(l) \equiv |\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{l})-\boldsymbol{v}(\boldsymbol{x})|$ of the velocity field on scale l:

$$\delta v(l) < cl|\ln l|. \tag{1}$$

It is natural to expect that the viscosity term gives further regularization effects in the Navier-Stokes equations, so that any model describing the enstrophy cascade must take (1) into account. The analogy of the β, or random β models (for a general discussion on β models see Paladin and Vulpiani (1987)) is therefore not useful for the direct enstrophy cascade in 2D (Rose and Sulem 1978, section 7.3). In contrast with the regularity bound (1), these fractal models give origin to singular velocity gradients in the limit of infinite Reynolds number. In fact, the β model of Shivamoggi (1990) gives

$$\delta v(l) \sim l^{1-(2-d_{\mathrm{F}})/3}$$

in a hypothetical active region with fractal dimension $d_{\mathrm{F}} < 2$ where the enstrophy dissipation concentrates.

In this journal (Benzi *et al* 1986), we have proposed a model with Markovian rules for the fragmentation through the cascade in order to avoid these unphysical singularities. Nevertheless this model, as well as the model of Shivamoggi, cannot give the scaling of the power spectrum.

This is our second major remark. From the structure functions, i.e. from

$$\langle \delta v(l)^2 \rangle \sim l^{\zeta} \tag{2}$$

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with $\zeta = 2 + (2 - d_F)/3$, Shivamoggi (1990) derives the spectral law for the energy spectrum

$$E(k) \sim k^{-\alpha} \qquad \text{with} \qquad \alpha = 1 + \zeta = 3 + (2 - d_F)/3 \tag{3}$$

using dimensional counting. Such a derivation would be correct only if $0 < \zeta < 2$, that is $1 < \alpha < 3$. On the other hand, from $\zeta \geqslant 2$, one only gets the bound $\alpha \geqslant 3$ (Babiano *et al* 1984).

Assuming a locality in the Fourier space and a constant transfer rate of enstrophy, Kraichnan (1967), Leith (1968) and Batchelor (1969) obtained the classical value $\alpha = 3$ in the marginal case $\zeta = 2$.

We believe that the origin of the steeper spectrum found in high resolution simulations (i.e. α varying between 3 and 5) cannot be explained by dimensional arguments, such as those used in β models, and that the cascade process could have a smaller degree of universality than in three dimensions. On the contrary, one can argue (Benzi *et al* 1990) that these corrections are related to the fractal dimension d_ω of isovorticity lines:

$$\alpha = (7 - 2d_\omega). \tag{4}$$

It is worth stressing that isovorticity lines have fractal dimension $d_\omega = 2$, if they densely fill the whole fluid. In this case, the spectrum is the same as predicted by dimensional arguments: $\alpha = 3$. When coherent structures dominate the dynamics, we expect $d_\omega \approx 1$, corresponding to an energy spectrum with a spectral slope steeper than -3. It is a matter of speculation whether different universality classes for d_ω exist according to the particular initial conditions considered.

We thus suggest that the intermittent corrections to the dimensional scaling law $E(k) \sim k^{-3}$ are related to the possibility that the isovorticity lines do not fill the fluid in the most random way, so that one has $d_\omega < 2$. The mechanism for intermittency in two dimensions appears very different from the three-dimensional case.

In conclusion our main objections to the β model proposed by Shivamoggi (1990) are:

(i) it violates the regularity bound (1);

(ii) it is not appropriate for the calculation of the scaling law of the energy spectrum because of non-locality of the interactions in the Fourier space.

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